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Delineating the Epistemological Trajectory of Learning Theories: Implications for Mathematics Teaching and Learning

The purpose of this paper is to delineate the trajectory of fundamental learning theories and the way these theories have impacted the teaching and learning of mathematics over more than half a century. We argue that a critical examination of the depiction of learning theories and their inherent implications for the teaching of mathematics afford an understanding of the hierarchical evolution of the field of mathematics education. Needless to say, examining various learning theories in the mathematics education context unpacks epistemological and ontological core issues underlying the teaching and learning of vital topics that are assumed to account for the changes in global economics and business in a STEM world.

Introduction

The quest to explore how people learn mathematics has been a perennial concern for decades. For some time now, philosophers and learning theorists have been incessantly searching for optimal conditions under which learning occurs. A tremendous bulk of contesting theories has evolved, thriving to explain the roles of the learner and the teacher when engaged in acts of learning and teaching. Notwithstanding the fact that many learning theories differ in their empirical manifestations, nonetheless the underlying epistemologies are unequivocally analogous.

At the turn of the 21st century, the necessity to prepare children for a rapidly changing world has been consistently reiterated in the literature (Csikszentmihalyi & Schneider, 2000; Cornell & Hartman, 2007; Drucker, 2010). With the tumultuous political and economic climate prevailing worldwide exacerbated by challenges emerging in the rapidly changing context of new technologies and globalization, there is an unprecedented need to capitalize on how best to educate future generations. A critical examination of the trajectories that various theories of learning have taken since the last century provides a clear explanation on the fundamental problems facing education locally and globally. In a highly dynamic and versatile world, unfolding the best practices particularly for teaching mathematics and sciences instigates a pressing need for consideration by educators and policy makers. For mathematics education, the push toward standards-based curricula governed by accountability and teaching effectiveness dogmas juxtaposed with policies to enact cutting edge educational interventions in an ever-changing economy command inevitable challenges that cannot be overlooked. The highly political dynamics that depict the way standards are drafted and sanctioned in terms of content and pedagogy are determining the way educational policies are institutionalized and eventually endorsed at the grass roots level.

Nevertheless, recent studies on student motivation, attitudes and self-efficacy have shown that unless students are cognitively and emotionally invested in what they are learning, little knowledge will be acquired (Friedman, 2006; Furner & Gonzalez-DeHass, 2011). Naturally, designing learning environments that can trigger students' natural curiosity and stimulate their interest will significantly impact their learning. Thompson and Thornton (2002) found that when students are intrinsically motivated they are eager to learn. Yair (2000) argues that, "students' interest in what they learn, and their sense of enjoyment while learning, are highly correlated with the outcomes of learning" (p. 193). Furner and Berman (2003) found that teachers need to do more in the way they teach math to address attitudes toward math and mathematics anxiety in the classroom.

However, research has suggested that motivation of adolescents decline as they progress through junior and senior high school (Eccles, Midgley, et al., 1993; Gonzalez, 2002; Hidi & Harackiewicz, 2000; Williams & Stockdale, 2004). Additionally, the value children place on many academic activities, particularly mathematics and their beliefs about the usefulness of school decline as they get older (Wigfield, Eccles, & Rodriquez, 1998). Today, students are more often described as "physically present but psychologically absent" and thus are less likely to actively and enthusiastically engage in learning. Recently, student lack of motivation in the mathematics classroom has been a critical national concern in light of efforts to improve STEM (Science, Technology, Engineering, and Mathematics) education in the United States. One concern in particular is that the pipeline of students entering STEM does not meet the current demand for future scientists and engineers. One of the reasons attributed to the attrition rate of students embarking on a degree in STEM is students' underperformance in high school mathematics and their inadequate preparation in rigorous mathematics content. Hence, addressing this national need requires research and development of the best pathways to remediate the teaching of mathematics and thus provide sufficient support to students in learning the content.

This leads us to the following questions: How did our understanding of how students learn develop in light of widened exposure to subsequent research efforts in educational and cognitive psychology? And what are the major learning theories that have impacted frequent paradigm shifts in the field of mathematics education?

The purpose of this paper is to delineate the trajectory of fundamental learning theories and the way these theories have impacted the teaching and learning of mathematics over more than half a century. We argue that a critical examination of the depiction of learning theories and their inherent implications for the teaching of mathematics affords an understanding of the hierarchical evolution of the field of mathematics education. Needless to say, examining various learning theories in the mathematics education context unpacks epistemological and ontological core issues underlying the teaching and learning of vital topics that are assumed to account for the changes in global economies.

Drawing on extensive literature related to learning theories that have emerged in the past two centuries, and examining critically possible formulations on how mathematical knowledge and problem solving activity can be constructed, we highlight important implications that, we argue, provide the most compelling explanation of how students learn mathematics. By virtually appealing to research on the cognitive as well as the social construction of mathematical knowledge, we explore various trends in teaching and learning practices proposed by each perspective and investigate their practical significance and implications in the mathematics classroom.

We begin with a brief overview of major "grand" theories of learning, namely those of Behaviorism, Cognitivism, Constructivism, and Social Constructivism. More concisely, we examine contributions of cognitive and socio-cognitive theorists who advanced learning epistemologies that transformed mainstream perspectives on learning mechanisms. We expand the argument further to discuss the implications of these theories with regards to addressing students' attitudes and motivation in the mathematics classroom. Finally, we provide an account of possible contributions of complexity science to the teaching and learning of mathematics.

Theories of Learning

A handful of research studies classified learning theories according to where these theories stand relative to four main categories: the nature of knowledge; existence of mental representations; causal relationship between mental relationships and behavior; and the origin of knowledge (Illeris, 2004; Pritchard, 2005). Based on the above-mentioned categories, Byrnes (2007) established three major groups that encompassed *Meta-Theoretical Belief Systems* (MTBS): 1) Behaviorism, Neobehaviorism, and Cognitivism; 2) Structuralism and Functionalism; and 3) Nativism, Empiricism, and Constructivism. Each of these groups comprises a spectrum of ideologies, perspectives and belief systems that, we believe, can potentially explain the systemic evolution and acquisition of knowledge.

Behaviorism, Neobehaviorism, and Cognitivism

Behaviorists, including Edward L. Thorndyke (1898), argue that learning is the acquisition of new behavior that can be manipulated by the environment and may be completely characterized in terms of stimuli and responses relations. Almost all behaviorists describe "knowledge" as simply a succession of stimulus-response chain acquired through conditioning. Behaviorists believe that learning is observable and is directly evidenced by a change in behavior. This theory has been criticized as being a theory of animal and human learning that only focuses on objectively observable behaviors that discounts mental activities (Tuckman, 1992). Attacked and refuted by radical behaviorists, in particular by Skinner (1938), behaviorism was abandoned in favor of operant conditioning where events in the environment determine and shape desired behavior (Post, 1988).Though this theory was prevalent from the 1950's through 1970's, it is still alive in the minds and practices of many educators in the 21st century.

In search for a more "humanistic" theory of learning, Neobehaviorism emerged calling for some mental mechanism that mediates between situations that elicit behaviors i.e. stimuli and specific behaviors i.e. responses (Tuckman, 1992). In his book *the conditions of learning*, Gagné, a leading Neobehaviorist, explains " the occurrence of learning is inferred from a difference in human being's performance as exhibited before and after being placed in a 'learning situation'" (1965, p.20).

When identifying the conditions necessary for learning to occur, Gagné (1965) cited five "categories of capabilities" which he defined as "conditions internal to the learner" (p.21). These capabilities include: intellectual, cognitive, verbal, motor and attitudes. While for Thorndike learning is one and only, Gagné (1965) spoke of eight varieties of learning, or external conditions each necessitating different capability and internal conditions, required from the learner. These types are seemingly cumulative and hierarchical such that each type is built on its prerequisite, the highest in the hierarchy being problem-solving or *type 8*. The hierarchy includes: *Problem-solving*; *Principle learning*; *Concept learning*; *Multiple discrimination learning*; *Verbal associations*; *Stimulus-response learning,* and *Signal learning*. Gagné also argued that for any learning to take place at any level special attention should be given to its prerequisites. Advocating the claims that the whole (goal or outcome) is the sum of its parts and that learning is rather specific and goal-directed, Gagné (1965) established task analysis as a technique that pinpoints conditions or behavioral objectives under which learning of a certain goal occurs. With a highlighted emphasis on "outcome content" (what to learn) rather than the process (how to learn), instruction was perceived to proceed from prerequisite skills to desired goals. Other important personal international factors such as motivation and establishment of attitudes and beliefs are ignored in favor of achieving behavioral objectives. Many drawbacks of Gagné's guided approach to learning have been frequently cited in the literature, such as limited transfer and neglecting other important forms of learning such as informal and discovery learning (Post, 1988; Tuckman, 1992).

On the other hand, theorists who have given more weight to cognitive explanations and mental structures are retained near the cognitive end of the continuum and are referred to as *Cognitivists* (Byrnes, 1992). Cognitivists such as Piaget, Bruner, and Dienes see knowledge as actively constructed by the learner in response to his/her interaction with the environment. In contrast to Neobehaviorists, Cognitivists emphasize the "how" of learning rather than the "what". In Cognitivists' views, the learner, an active part in the learning process, uses internal mental structures to organize, transform and retrieve information when acting on a learning environment (Chahine, under review). As such, learning is intrinsic, holistic and personal to the individual and can be enhanced through interactions with others and with proper physical materials in the environment. A closer look at the works of these theorists merits our discussion of learning theories.

Piaget's Developmental Theory

A prominent Cognitivist in the 20th century is the Swiss biologist and psychologist Jean Piaget (1896-1980). Piaget is renowned for constructing a highly influential model of intellectual development and learning. Piaget's developmental theory is based on the idea that the child builds cognitive mental structures in an attempt to make sense of his or her environment (Piaget, 1952). These stages, at least for Swiss school children, increase in sophistication with development and include: *Sensorimotor stage (birth - 2 years old)* where the child, through physical interaction with his or her environment, builds a set of concepts about reality and how it works. *Preoperational stage (ages 2-7)* in which the child is not yet capable of abstract conceptualization and therefore needs exposure to concrete physical situations. The primary deficiency at this stage is what Piaget (1952) calls the *reversibility principle* where the child cannot grasp the idea of conservation of quantity. *Concrete operations (ages 7-11)* where physical experience accumulates, the child starts to conceptualize, creating logical structures that explain his or her physical experiences. He further claims that abstract problem-solving is also possible at this stage. Although a child's actions are internalized and reversible, Piaget (1952) explains that the child cannot deal with possibilities that are outside the realm of his/her direct experience, and finally *formal operations (beginning at ages 11-15)* where the child is capable of constructing formal operational definitions and thus, abstract concepts

One of the significant implications of Piaget's developmental theory is related to teaching basic concepts. A direct assumption is that a child should be helped to progressively proceed from the concrete to the more abstract modes of thought. Much of the mathematics taught following traditional curricula contradicts this assumption, through teaching by telling formal explanations of concepts are presented to the child in a different mode of thought than his own.

In adapting to his/her environment thus avoiding and minimizing mathematics anxiety, Piaget (1977) contends that individuals use two mechanisms: *Assimilation and accommodation*. *Assimilation*, which involves incorporating new ideas into existing schemata in Piagetian theory, is somehow similar to the Behaviorists' concept of stimulus where having learned to respond to one stimulus makes it easy to respond to another similar stimulus. *Accommodation*, on the other hand, involves modifying existing schemata to fit the newly assimilated information. Striking a balance or equilibrium between both assimilation and accommodation is the basis of intellectual development. Additionally, Piaget (1952) views intelligence as "adaptation to new circumstances" (p.151) and explains that in any intelligent act " the need which serves as motive power not only consists in repeating , but in adapting , that is to say, in assimilating a new situation to old schemata and in accommodating these schemata to new circumstances" (Piaget, 1952, p. 182). Thus, Piaget does not view intelligence in terms of content or amount of knowledge, but rather as an arrangement or structure and a way in which information is organized. Other developmental factors that contribute to cognitive development include maturation, active experience social interaction, and equilibration or self-regulation.

A basic tenet of learning by discovery or exploration requires a vigorous exposure to and involvement with the environment. A direct implication of Piaget's theory is the focus on development of *schemata* that potentially facilitates problem-solving. The developmental stages delineated by Piaget are highly relevant to the teaching and learning of mathematical concepts. For example, at the primary level, students can be taught conservation problems in late elementary and early middle level through tasks involving seriation and classification. Moreover, structuring the physical environment by making multiple learning centers where students can be actively and purposefully involved in the learning process is necessary to enhance students' attitudes toward math and motivation. In addition, utilizing hands-on activities with varieties of manipulatives and multiple physical embodiments helps children learn operations appropriate to their level of development. Piaget's theory calls for more emphasis on integrative themes, like probability and statistics in early grades as well as inclusion of basic algebra concepts in primary grades, the case we see now in *Common Core State Standards* (National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO) (2010). Engaging in hand-on explorations focused on these topics reinforces students' beliefs in the relevance of mathematics to real life situations thus fostering positive math attitudes and motivation for further learning.

Zoltan Dienes' Theory

One of the significant pioneers who established a theory specifically directed towards understanding the learning of mathematics is Zoltan Dienes. Dienes' theory of learning encompasses four major principles: *The Dynamic Principle; the Perceptual Variability Principle; the Mathematical Variability Principle; and the Constructivity Principle* (Dienes, 1960). In delineating the "skeleton" of his theory, Dienes (1960) acknowledges the works of Piaget, Bruner and Sir Frederick Bartlett whose ideas resonate within each component of this theory. *The Dynamic Principle* outlines three basic stages for concept formation, each requiring a different kind of learning: First, free *play* stage which requires free unstructured, but rather purposive activities that allows open and informal experimentation with the task at hand; second, *concept realization* stage where the child is exposed to varying experiences which are "structurally similar (isomorphic) to the concepts to be learned" (Post, 1988, p.7). Finally, the third stage represents the development of the math concept and sufficiently applying it to varied contexts. Dienes (1960) calls these stages as "stages of growth necessary before a mathematical predicate or concept becomes fully operational" (p.42). He also argues that learning a mathematics concept necessitates a clear understanding of a set of variables underlying this concept as well as other factors that are extrinsic to it and which are seemingly embedded in the experiences provided. This calls upon his second and third principles, *the Perceptual Variability Principle* and *the Mathematical Variability Principle.* In *the Perceptual Variability Principle,* Dienes (1960) maintains that conceptual learning is enhanced by exposing the child to multiple, varied physical representations on the same concept. This allows the child to *abstract* similar elements underlying the different embodiments. However, in the *Mathematical Variability Principle,* Dienes suggests that a concept dissected into constituent sub-concepts can be *generalized* when "all possible variables [are] made to vary while keeping the concept intact" (p.42). Finally, in the *Constructivity Principle,* Dienes explains the mode of thought involved in concept construction. When discussing the structure of a task appropriate for the child's thinking, Dienes (1960) identifies two levels of "logical complexity": *constructive* and *analytical*. He further argues that children, when young, are involved in constructive types of thinking where they are actively engaged in episodes of free play. Having had the proper opportunities to think constructively, Dienes asserts that the child at, around 12 years of age, is then capable of more analytical mode of thinking and thus well prepared to decode and analyze tasks using his initial constructions.

Other important implications of Dienes' work to increase motivation and invoke positive attitudes in the mathematics classroom involve re-structuring the environment to include a variety of manipulative and learning tools, encouraging group work, and reinforcing the role of the teacher as a coach and facilitator.

Structuralism and Functionalism

Founded by the British psychologist Edward Tichener the theory of Structuralism capitalizes on the role of consciousness and introspection in describing numerous cognitive processes and mental structures including sensations, images and affections.

Structuralists contend that the sum total of mental structures and their interactions comprise the conscious experience. A major implication of this theory is the role that immediate experiences and reflection play in stimulating complex perceptions and bolster students' positive attitudes toward math thus invoking learning (Carlson, 2010). Structuralism as a movement lost its popularity in the 1960s with the emergence of post-structuralism, a French movement that critiqued the basic tenets of structuralism and called for social constructionism as a means to expose "subjugated knowledges" (Foucault, 2003).

In principle, Structuralism and Functionalism symbolize two end-poles of a continuum vis à vis the degree of emphasis each paradigm merits either conceptual or procedural knowledge. Rittle-Johnson, Siegler, and Alibali (2001) defined *procedural knowledge* as "…the ability to execute action sequences to solve problems" (p. 346). They also argued that procedural knowledge involved mainly the use of previously learned step-by-step techniques and algorithms to solve specific types of problems. Furthermore, the authors explained that *conceptual knowledge* entails "… implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain" (p.346).

A handful of research studies support the hypothesis that forming correct problem representations is one mechanism linking improved conceptual knowledge to improved procedural knowledge (Jitendra, 2002; Hoffman & Spatariu, 2008). Rittle-Johnson, Siegler, and Alibali (2001) defined problem representation as "the internal depiction or re-creation of a problem in working memory during problem-solving" (p. 348). Furthermore, a number of studies investigated the role that problem representation played in changing performance either positively or negatively, depending on the circumstances (Kohl & Finkelstein, 2005).

While proponents of structuralism capitalize on the nature and organization of concepts, functionalists focus on how the mind *operates* in the course of problem-solving and information processing (Byrnes, 2007). Millroy (1992) defined the Piagetian approach as "the structural developmental" approach for it specifically concentrates on individual cognitive development, with little emphasis on social influences.

Problem Solving Theory

Most of the approaches to problem solving established during the past thirty years practically fall under one of the two contradictory treatments: The information processing model of human thinking (Newell & Simon, 1972) and the social practice theory (Turner, 1982). The information processing model aims at describing the general processes of problem-solving, minimizing the role of individuals in the problem-solving process (Putnam et al., 1989). Schoenfeld (1983) extended by far the scope of traditional conception of mathematical problem- solving highlighting four fundamental dimensions of good practice: resources of mathematical knowledge, heuristic strategies, control over the process of working on problems, and a deep understanding of the nature of mathematical argumentation.

Current interest in problem-solving as a "practice" reflects a trend in which learners are characterized as more active and where problem-solving is viewed as a series of activities (Lave et al., 1990). As a matter of fact, it has been widely argued that emphasizing the value of teaching problem solving in schools as a mathematical practice in contrast to rote procedural approaches enables the learner to gain more in-depth understanding of mathematical principles underlying such practice (Nunes et al., 1993). The demand to integrate the learner with the surrounding environment, practice and culture is an approach that is embraced by the social practice theorists. In principle, the social practice theory seeks to interpret the meaning of social activity in a number of environments and to discern their causes.

A close akin of the social practice theory is the Activity theory. Activity theory has been widely employed in socio-cultural and socio-historical research that aimed to scrutinize human activity systems (Jonassen & Murphy, 1999). It has been extensively used as a framework for developing Constructivist Learning Environments (CLE) and cognitive tools for learning. The assumptions of activity theory resonate with those of constructivism, situated learning, distributed cognitions, case-based reasoning, social cognition, and everyday cognition.

Activity theory has its roots in the Soviet cultural-historical psychology of Vygotsky, Leont'ev (1978) and it represents an alternative perspective to the claim that learning must precede activity. The most fundamental assumption of activity theory is that the "human mind emerges and exists as a special component of interactions with the environment, so activity (sensory, mental, and physical) is a precursor to learning" (p.64).

Although the learner has a central role in defining activity, very little, if any, meaningful activity is accomplished individually. Activity theory contends that learning and doing are inseparable, and that they are initiated by intention (Jurdak & Shahin, 2001).

Activity always involves an ensemble of artifacts (instruments, signs, procedures, machines, methods, laws, and forms of work organization) contrary to cognitive psychologist focus on mental representations. Jonassen and Murphy (1999) argue that Activity theory presents a new perspective for analyzing learning processes and outcomes for designing instruction. He adds: "rather than focusing on knowledge states, it focuses on the activities in which people are engaged, the nature of the tools they use in those activities, the social and contextual relationships among the collaborators in those activities, the goals and intentions of those activities, and the objects or outcomes of those activities" (p. 68).

Nativism, Empiricism, and Constructivism

Theories of cognitive development are situated under three fundamental views related to the origin of knowledge: The empiricist view, the nativist view, and the constructivist view (Saxe, 1991). Byrnes (2007) argues that all nativists share a belief that knowledge is innate. While the empiricist view favors the position that the environment is the source of knowledge, the nativists advocate the need of knowledge structures to organize and categorize experience (Saxe, 1991). The polar opposite of nativism is *Empiricism*. In this view, it is believed that individuals possess no *a priori* knowledge but rather that most knowledge is perceived to be acquired through exposure to the world.

Constructivists, on the other hand, adopt the premise that knowledge is not inherent in the human mind nor in the environment, but is rather actively constructed by the individual as a result of his/her interaction with the social and physical environment (Millroy, 1999). As a matter of fact, constructivism derives most of its ideas from Piaget's theory of cognitive development.

Partly inspired by Piagetian ideas, constructivism goes further to emphasize the role of others in the construction process. Through negotiation and communication with others, constructivists claim that people receive continuous feedback as well as agreement concerning their personal constructions (Brooks & Brooks, 1993). In this respect, constructivism agrees with Vygotsky's theory which asserts that cognitive functioning occurs first on the social level, between people, and that the child afterwards internalizes this in his development (Vygotsky, 1962).

Lev Vygotsky and Socio-Cultural Theory

Vygotsky was famous for introducing the term "the zone of proximal development" which he defines as "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1962, p.103). Through scaffolding, an adult can facilitate and adjust the environment to enhance students' positive attitudes toward math and maximize learning. In a sense ZPD represents for Vygotsky the social context in which learning takes place. In this respect, individual learning is seen as inherently guided by the social world through the introduction of society's tools and by engaging with more experienced members of society. As a socio-cultural constructivist, Vygotsky adopts the *social cognition learning mode*, which asserts that individual cognition is socially and culturally mediated (Vygotsky, 1962). Like most of the social cognition learning theorists, Vygotsky (1962) believed that culture teaches children both what to think and how to think. To Vygotsky language is a primary form of interaction through which adults transmit to the child the rich body of knowledge that exists in the culture.This social context shapes the range of potential each student has for learning. As learning progresses, the child's own language comes to serve as a primary tool of intellectual adaptation (Vygotsky, 1962). Therefore, Vygotsky (1962) viewed the process of learning as mainly an internalization of a body of knowledge and tools of thought that first exist outside the child.

As a result of his intense emphasis on the social dimension of learning, Vygotsky's view inherently diverges from that of Piaget. Kincheloe (2004) argues that Vygotsky was critical of the way Piaget investigated children's cognitive abilities while working alone. He contends that a true measure of individual ability is only revealed through collective social interactions. Kincheloe (2004) also explains that, with his notion of learning as dependent on ZPD, Vygotsky discards Piaget's concept of development as a systematic shift from one discrete stage into another and highlights the role that artifacts i.e., sign systems, play in developing cognition.

Researchers influenced by Vygotsky have basically emphasized the role of cultural practices in analyzing the relation between culture and cognition (Millroy, 1992). Their investigations have generally focused on studying people's use of math outside the classroom (Gay & Cole, 1967; Jurdak & Shahin, 2001; Scribner, 1986; Lave, 1988; Lave, Smith, & Butler, 1990; Saxe, 1991, Millroy, 1992). The practice of math has been explored in the contexts of everyday activities. Two main groups of researchers have explored the use of math in settings outside school: those interested in "everyday cognition" or "cognition in practice", where Lave is a prominent figure, and those interested in "ethnomathematics", where D'Ambrosio (1985) is a key figure. Both groups of researchers call for a new conceptualization of mathematics that is rooted in nonacademic practices. The work of these groups focuses on three main issues: Analysis of school practice, investigation of the transfer of school knowledge to out-of-school situations, and using the social theory of practice to challenge conventional cognitive theory.

Jerome Bruner's Representation Theory

With the hypothesis that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (Bruner, 1960, p.47), Bruner envisioned learning as successively proceeding through three hierarchical stages. He held the view that experience is coded and processed in such a way to ensure its retrieval when needed. Such a coding system Bruner calls *representation.* He converted his ideas about modes of representation into chronologically structured stages of development and claimed that understanding in any domain must involve three modes of representation: *enactive* through *habits of acting*, *iconic* in the form of pictorial images and *symbolic* through written symbols like language. "By enactive, I mean a mode of representing past events through appropriate motor response" (Bruner, 1964, p. 69). In this regard, he agrees with Vygotsky's notion of ZPD in helping children understand and master a concept by proceeding from physical practical actions then using imagery and pictorial representation after which written symbols can be used.

Not surprisingly, in his more recent work, Bruner expanded his theoretical framework to encompass the philosophical and sociocultural aspects of learning. In his approach to instruction, Bruner emphasized four major ideas: structure of the discipline, readiness to learn which depends on rich learning experiences and enthusiastic teachers, intuition, and motivation. He also calls for curriculum to be organized in a spiral manner so that the student continually builds upon what he/she already learned.

Perhaps one of the most important implications of Bruner's work is the significant role that different modes of representation play in learning mathematics. Cramer (2003) argues that experiencing the benefits of using multiple representations as well as allowing possible translations among these different modes of representations helps teachers become more aware of the weaknesses inherent in any curriculum and thus respond by incrementing it with outside resources.

In addition to the significant role that external representations play in students' active construction of mathematical concepts, many studies have investigated the impact of students' beliefs, feelings and affective representation in enhancing or hindering mathematical understanding in the classroom. Goldin (2003) calls for "a good balance between the standard manipulation of formal notational systems… and the development of other representational modes: imagistic thinking, involving visualization; visual imagery, pattern recognition, and analogical reasoning; heuristic planning, involving diverse problem solving strategies; and affective representation" (p.283). By the same token, Monk (2003) highlights the importance of providing concrete, multiple embodiments that are meaningful to students, and further explains "the goal is not to select one or two representational forms for students to learn and use in all situations but, rather, to teach students to adapt representations to a particular context and purpose and even to use several representations at the same time.

This goal represents a shift from representation to representing" (p.260).

Beyond Theories: Epistemological Frameworks

As has been noted by several theorists (Byrnes, 2007, Lesh & Doerr, 2003), perhaps there no longer exist "grand theories" such as Piaget's which attempt to explain many aspects of cognition. Instead we have many "micro-theories" or what Lesh and Doerr (2003) calls "models" designed to account for a "specific purposes in specific situations" (p.526).

An increasing number of theorists are developing numerous epistemological frameworks by combining constructs from several theories which may potentially contribute to our understanding of how mathematics is learned. The current trend in mathematics education research is converging towards adopting a *models and modeling perspective* for the teaching and learning of mathematics (Lesh & Doerr, 2003). The main gist underlying this call is not only to progress an agenda of research that builds on recommendations of constructivism, but also to include "a wealth of recent advances in fields of mathematics ranging from complexity theory to game theory- where a variety of different types of systems thinking tends to be highlighted." (p.555)

Complexity Science

Recent research in complexity science has revolutionized the conduct and method of science by revealing new perspectives and possibilities that challenge beliefs and ideas of contemporary learning theories. The exponential advancement of innovations in digital technologies is pushing the boundaries of learning beyond existing traditional classrooms by creating learning environments that seamlessly capture and model complexity in the world. Such technologically-supported environments are transforming education and expanding the concept of schooling beyond school settings. In an article entitled, *Understanding Learning Systems: Mathematics Education and Complexity Science*, Davies and Simmt (2003) describes mathematics classes as " adaptive and self-organizing complex systems" where "learning is understood in terms of ongoing, recursively elaborative adaptations through which systems maintain their coherences within their dynamic circumstances" (p.138).

While complexity scientists agree with the views advocated by situated learning theories and social constructivism where the emphasis is placed on analyzing the dynamics of emerging experiences in context, however they transcend individual and social constructivism in calling for a conceptual shift away from mathematics as content and toward "*emergent terms*" (Davies & Sumara, 2007). This perspective is concurrent with the view that mathematics is socially and culturally constructed.

Within the realm of complexity science and where the world is seen as becoming increasingly complex, the discrete distinction between *teaching mathematics* and *teaching children* collapses in favor of a rather nested, integrated whole whose entities are inseparable. In this context, Davies and Simmt (2003) explain: "To teach children well, we argue, we must conceive of our activity in terms of active participation in the body of mathematics knowledge by creating the conditions for the emergence of bottom-up, locally controlled, collective learning systems" (p. 163).

Furthermore, the authors emphasize five basic conditions that help facilitate the creation of a classroom community: a) *internal diversity* which, calls for respecting and supporting individual differences in the classroom; b) *redundancy*, which shapes the criteria for negotiation and participation in classroom activities; c) *decentralized control* where less emphasis is given to the role of the teacher as expositor and calls for more thought-provoking activities; d) *organized randomness* where purpose is given to each activity; and e) *neighbor interactions* where ideas are productively negotiated and provoked.

To this end, we argue that building a mathematical community that supports and nurtures individual students' understanding of mathematical concepts is of the utmost importance. Such a *classroom collective* represents a medium where meanings are negotiated and shared and where instruction is captured in episodes of "teachable moments". These contemporary views present a shift of emphasis from individual forms of learning to a rather social, collective mathematical knowledge developed as a result of a network of interactions in the context of the classroom.

Impact of Learning Theories: What lessons can teachers learn?

Much of the current debate about standards in mathematics education arose from opposing views about how people learn mathematics. The questions: Should automaticity and quick recall of facts with emphasis on procedural skills precede problem-solving? Or should reasoning and constructive thinking reign in the mathematics classroom even before skill development? These questions remain incessantly unanswered.

Arguments initiated by national standards have been loud for some time. Since the release of *A Nation at Risk report* by the National Center of Education Evaluation (NCEE) in 1983, many organizations, including NCTM, have published documents that delineated goals and standards for mathematical content and processes for grades K-12. In 1989, NCTM published *Curriculum and Evaluation Standards for School Mathematics* expanding on the recommendations of the *Agenda for Action* publication issued in 1980 and provided a "road map" for states and school districts in developing their curriculum guidelines. *Principles and Standards for School Mathematics* (NCTM, 2000) followed, building on the preceding publication and adding "underlying principles" for school mathematics. In spring 2010, The National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO) released the National *Common Core Math Standards* that specifically concentrates on fundamental shifts in content and pedagogy(National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO) (2010)). Teachers don't exactly feel prepared to manage the bridging required to access quality instructional resources and teaching materials that will aid in faithful implementation of the standards for students' academic success (Schmidt, Houang, & Cogan, 2011). As Confrey and Stohl (2004) explain: "A successful curriculum is impossible if it does not pay attention to the abilities and needs of teachers" (p.92). What seems to be essential is how teachers translate what they know about learning from these theories into practical everyday applications in their design and delivery of instruction. In a chapter entitled *Beyond Constructivism: An Improved Fitness Metaphor for the Acquisition of Mathematical Knowledge*, Lamon (2003) explains: "classroom interpretations of constructivism are not necessarily headed in a useful direction and that before the pendulum of reform sweeps too far to the right, it may be time to consider alternative, but not necessarily competing, perspectives on the development of mathematical knowledge" (p. 436). With so many voices contributing recommendations and standards, teachers are faced with overwhelming challenges in this STEM world we live in. Interestingly enough, while so many considerations are directed toward "what" mathematics students should learn, only a dim initiative has been advanced on "how" children should learn.

The impact that theories of learning can have for the teacher in the mathematics classroom is far more complex than what it seems to be. Lamon (2003) argues that the powerful recommendations set by Piaget's cognitive constructivist view, as well as Vygotsky's sociocultural perspective of acculturation and negotiation can be less effective if misinterpreted and overgeneralized in the mathematics classroom. Regardless of their graduate training or experience, we argue that teachers bring to their practice what might be called "personal theories" of teaching and learning. These theories may be only partly conscious; however these epistemologies are what guide teachers' everyday decisions about planning, subject content, and classroom behavior. "Personal theories" of teaching and learning grow out of our experiences as students and teachers and begin developing while we are children. We develop predispositions toward certain learning "styles", just as we gravitate toward certain teaching styles. If we are to trace the literature on best practices for improvement of teaching a useful focal point emerge which is an analysis of the degree to which teachers' practice is consistent with their introspection about teaching-their "personal theory."

Summary

The 21st century is an exciting and challenging time for mathematics teachers as the opportunities to expand teaching and learning are becoming more and more pervasive in formal and informal education. Associated with this growth are the increasing number of demands and expectations behooved on teachers to serve as leaders enacting cutting-edge instructional practices in their classrooms. Our world, through the use of complex satellite systems, is connected with an invisible digital network that makes todays' classrooms inevitably global. Students now learn from a multitude of resources that range from textbooks to live videoconferences with people geographically separated by thousands of miles. The world is becoming more open to students through live, streamed videos enabling them to see the world "as it happens" without any controls. In this climate, teachers are expected to be well-versed in the newest learning technologies and products in order to best prepare students for the global digital workforce. However, in the midst of increased technology, access to resources and to professional development in today's highly diverse schools remains the fundamental quandary dodging opportunities for improvement in teaching and learning (Smaldino, Lowther, Russell, 2012). While teachers are called upon to expand their professional knowledge and growth by staying informed of new technologies that have positive impact on students learning, very little effort is advanced to increase teacher capability to use assistive technologies to facilitate student success. With heightened tension from policy makers, the public as well as educational media, teachers find themselves pressured to deliver quality math instruction to ensure that all students achieve academic proficiency in a STEM world (Chahine & King, 2012).

On February 17, 2009, President Obama signed the American Recovery and Reinvestment Act (ARRA), spurring the Race to the Top (RTT) initiative, which was a way to invest in the nation's education system in an effort to reform schools. President Obama declared: "America will not succeed in the 21st century unless we do a far better job of educating our sons and daughters…The race starts today" (Bosser, 2012, p.1). RTT focuses on more rigorous standards and closer evaluation of teachers whose performance is inextricably linked to students' achievement. However, to empower teachers toward better teaching performance and thereby increase student academic progress, perhaps we need to provide more incentives for teachers to undertake the challenge of educating future generations.

In this critical educational climate, it's reasonable to argue that what mathematics teachers need more than ever, is sincere support and collaboration that is free from any political obligation. Instead of focusing on issues of accountability and fueling efforts towards more exam–driven instruction, perhaps teachers need more support to understand how students think, to uncover cultural and social backgrounds that help interpret their ways of thinking, and to capitalize on those practices that maintain equity, diversity and high quality instruction for all students so that more student are seeing success with math and liking it too.

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