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What Concept of Mathematical Proof Should Future Teachers Possess?

This article argues that teachers who teach at any grade level need to possess a sophisticated and flexible notion of mathematical proof. In particular, teachers must be able to teach the concept of proof in a variety of ways depending on their students' mental development. Rigorous and intuitive proof structures are demonstrated. Mathematical proof is discussed as a way to engage students in a body of knowledge that is certain and provable, a quality absent in virtually all other disciplines.

Mathematics at the K-12 level is increasingly being thought of as something to understand, instead of merely a set of procedures used to solve problems. If a student or teacher is to really understand mathematics, he/she must enter into the reasoning behind algorithms, move beyond special cases to general principles, and be able to persuade himself/herself (and others) why and how mathematical postulates work (NCTM, 2000).

Some educators would be satisfied to have their students' understanding rest upon logical-deductive proof structures. One example of a logical-deductive proof structure is the rigorous "two-column" proof of high school geometry. Another example which is often found in collegiate mathematics courses, or high school textbooks, is a structure of if-then statements seamlessly adjoined together which leads from a premise to a conclusion.

Other educators would be satisfied if their students had a more intuitive feel for why things in mathematics are true. For example, when trying to decide if an odd plus an odd is always an even, one might take a certain workable example, say 9 + 5, and notice that you can take 1 away from the 9 and likewise from the 5 and you have 8 + 4 + 2 = 14, an even. It seems believable to many people that this process would generalize to any case, and thus the statement odd + odd = even for all whole numbers, would be considered true.

Which approach toward proving mathematical truths should be promoted in the education of future K-12 teachers? By extension, what view of mathematical proof would we like these teachers to convey toward their future students? This paper argues that future K-12 teachers should be acquainted with a variety of proof structures, from the more rigorous, formal proof structures, all the way down to intuitive, heuristic styles of proof. The reasons for this stance are: (1) proofs are intended to establish certain truth claims, and thus, the structure of a proof must be flexible, dependent upon whom the proof is intended to persuade; (2) teachers must understand that their future students are limited developmentally in conceptualizing mathematics in the abstract; and (3) teachers who learn that mathematics is an exercise in finding out what things are true, and that *this truth is able to be verified in an absolute sense*, are learning a mathematics which is engaging and fulfilling. A future teacher who is able to persuade themselves and others of mathematical truth, knows what proof techniques are effective at different age levels, and models mathematical engagement and fulfillment, will be an effective teacher.

In an ideal world, teachers from kindergarten through high school would first convince themselves that a mathematical proposition is true, and then be able to persuade their own students with arguments which seem convincing to the students. These teachers would also convey to students the importance of convincing others (such as other students, teachers, and CSAP or AP Calculus test graders) of the truth or falsity of mathematical propositions. For some audiences, certain formal proof structures and if-then arguments are the most persuasive. In other cases, giving a reasoned argument in lay men's terms is much more acceptable. To teach effectively, future teachers should enable students, as the Principles and Standards (NCTM, 2000) state, "to select and use various types of reasoning and methods of proof" (p. 402).

One example of a mathematical statement that can be proved in various ways is as follows. Clearly, 9^2 in base 10 is 81, and 8^2 in base 9 is 71. Suppose we claim that (a-1)^2 in base a is always the number composed of two digits: a-2 and 1. What can be done to convince a person that 5^2 is going to be 41 without actually computing the product? In an elementary classroom, computing several products of nearby bases (e.g. 3^2 base 4, 4^2 base 5, 6^2 base 6, etc.) and noticing the pattern might be quite convincing to the average student. Such an approach is labeled by Balacheff (1988) as "naïve empiricism" (p. 222). However, by high school, students have hopefully been exposed to mathematical statements whose truth goes against the intuition, even though they work for almost all test values. For example, as the perimeter of a polygon increases, the area does not necessarily increase (Ma, 1999). A wary student might not be convinced that just because 9^2 in base 10 is 81, and 8^2 in base 9 is 71, that this assures 5^2 base 6 will be 41, let alone all values for a. Perhaps base 6 is a special case. What would convince such a student? There are at least two approaches which might be demonstrated. The first, a "generic example" approach (Balacheff, 1988), would be to randomly choose a base, say 21, and then calculate 20^2. If the answer confirms our pattern, most students would be convinced that the pattern extends to every case, in particular 5^2 base 6. This differs from the elementary school argument in that the example is random, and somewhat representative of the *class* of items we are trying to test. For example, suppose I make the claim that 5 plus anything is less than 100. If this hypothesis is tested with 5 + 6, 5 + 7, …5 + 25, many examples have been worked and it seems true. However, if a "tester" case is taken, exploiting the *class* of objects 5 + n, an extreme n would be randomly chosen, say 15348, and then the claim would be false.

Nevertheless, there remains a second approach to the base 6 problem which would be sufficient to convince any mathematics professor, namely, the more rigorous symbolic approach. Let (x-1) be a base x number. Then (x-1)^2 = x^2 – 2x +1 = x(x-1) + 1 = (x-1)x + 1 = [(x-1) groups of x] + [1 group of 1]. And applying the generalization to our base 6 case, we have [5 groups of 6] + 1 which is, by definition, 51 base 6.

Not everyone agrees with the aforementioned view. Many teachers, even in high school, relegate proof to one class—ninth grade geometry (Moore, 1994). Other, more idealistic mathematicians, are nervous when proofs are diluted by reliance on technology (Kleiner & Movshovitx-Hadar, 1997, p. 22) or contextualized by a community of learners. Some picture mathematics as an axiomatic system disjoint from a community of learners and a proof is a proof because it follows from axioms—it does not need to be designed to persuade different audiences (See, for example, Howell & Bradely, 2001, chapter 2).

If future teachers were comfortable with a wide variety of proof techniques and structures, they would still need to know at what age levels students can grasp abstractions, and what to expect of students at various stages in their development when they attempt to prove mathematical conjectures. Balacheff (1988) concludes that a proof hierarchy exists among students of the same age; that is, students engage with mathematical proof at different conceptual levels, from testing specific cases to conceptually grounded "thought experiments." The assumption of such a hierarchy must be even more pronounced across different age levels. Piaget's premise that children pass through an invariant sequence of four stages of cognitive development (Driskol, 2000, p. 191) implies that students, especially ages 2-11, are cognitively limited (or predisposed) towards certain types of reasoning. For example, according to Piaget, a student in the fourth grade would have difficulty working in hypothetical situations (e.g. Assume that even + even = odd and show a contradiction) but can logically reason using concrete objects (e.g. Prove that the sum of the legs of the chairs in the classroom will always be a multiple of four).

Teachers must also keep in mind that proving things in mathematics is different than proving someone is guilty in a court of law, or deciding that Shampoo A is of a better quality than Shampoo B. As Yackel and Hanna (2003) assert, "By its very nature, mathematical proof is highly sophisticated and seems to be much more challenging intellectually than many other parts of the school mathematics curriculum. To a large extent this is so because the kind of reasoning required in mathematical proof is very different from that required in everyday life" (p. 231).

Yackel and Hanna go on to suggest several reasons why students aren't always impressed with proofs or understand why a proof is necessary. For example, the laws of inference in proofs are often anti-intuitive (false implies false is true). Or sometimes, the theorem is so intuitively obvious that students fail to see the need to prove it. Piaget suggests that a second-grade student would have great difficulty proving that two rows of three tiles covers the same amount of space as one row of six tiles because the student is in the "preoperational period" of his/her life, and tends to see only one aspect of most problems. A second-grade student sees the 2x3 tiles as different from the 1x6 tiles because they look different, and it would never occur to him/her that they cover the same area because they are two representations of the same number of tiles.

Not all educators agree that younger students are limited in their proof ability as much as Piaget would suggest. The Principles and Standards writers argue that proving mathematical theorems in high school is so difficult for students because they have never been exposed to proof structures from the earliest grades. Additionally, Lampert (1990) reports on an elementary classroom experience in which students attempted to find a pattern in the ending digit for the sequence: 5^4, 6^4, 7^4, etc. By the end of the lesson, each student was able to state with conviction what he/she thought the pattern was, provide a "proof" that the pattern could continue, or explain in his/her own words another student's assertion. This suggests that under the right conditions, students of all ages can engage in quality proof work.

Many students, including future K-12 teachers who are currently students, see a theorem in the text and use it, ignoring the proof. They learn that an integer is divisible by 9 if the digits sum to 9, but would never think to ask why this is, or even wonder if they have been lied to or misinformed. Such mathematics learning, in my opinion, is unfulfilling, and is not very engaging for the learner. If mathematics is framed as an endeavor to prove the true or "falseness" of mathematical statements, mathematics becomes an exploration with a purpose, with unexpected outcomes and chances to make new discoveries. It is engaging. The well known mathematician Bertrand Russell, when expressing his feeling toward the discipline noted:

"Mathematics possesses not only truth but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, sublimely pure and capable of a stern perfection such as only the greatest art can show" (Russell, 1907, as quoted by Howell & Bradely, 2001, p. 231).

Mathematics can become personally meaningful for teachers, as it was for Russell, and I believe that part of the attraction of mathematics is its “cold and austere” beauty and truthfulness. Furthermore, one of the characteristics almost unique to mathematics is that virtually everything can be verified as true or false in an absolute sense (compare the claim, “even + even = even”, with the claim “smoking causes lung cancer”). A proof gives fulfillment to the learner. A learner does not have to check every case, can depend on the result in the future, and can solve problems with confidence knowing that the theorems he/she references in solving a problem make perfect sense to him/her (Geisendorfer, 2006).

Many people do not agree that mathematics needs to be engaging or fulfilling in the sense described above. Some would argue that mathematics is something we do, therefore as long as we know that the quadratic formula works (and we know it does because it is in the textbook), then getting to know how to use it is really what is important. The underlying rationale or proof of it is not really important. This viewpoint is often associated with the “back-to-the-basics” movement of the 1970’s, and dominates certain segments of public opinion even today. Also, timed tests with computations galore are often the norm both at the high school and college level, and thus one could argue that students need to be rapid computers of answers, as opposed to slow mathematical thinkers. A good example of this perspective of mathematical learning is seen in the Saxon textbook series.

In arguing this thesis, the assumption is made that teachers who learn and are comfortable with various proof structures will be able to (1) transmit these skills to their students, and (2) be able to persuade their own students of mathematical truths. To support the first assumption, I point out that teacher conceptual knowledge has been linked to student skills and achievement in various studies—for example, the Chinese and U.S. teachers examined by Ma (1999). To support the second claim, I refer the reader to Hanna’s (1989) research study on mathematical proof which suggests many students can be persuaded by *proofs that explain* when teachers prove things with the conviction that mathematics does make sense.

This paper has also made the assumption that if a teacher finds mathematics engaging and fulfilling, this will somehow transmit to their students. To support this, I mention Pehkonen and Torner’s (1999) article which suggests that mathematics educators significantly shape the view their students have of mathematics as a discipline.

In summary, teachers who know various proof strategies and use them in the classroom will be more effective in teaching students mathematics. My recommendation is the *gradual* shifting of K-12 mathematics curriculum toward reasoning and proof by district curriculum personnel. The word “gradual” is used because schools and teachers do not have the training, support, or background knowledge to radically shift away from society’s current views of mathematics and how it should be taught and learned. It is suggested that as proof becomes more and more a part of preservice teacher courses, and that over the course of the next 50 years, these gradual changes in the teachers will also amount to significant changes in schools and students.

More will have to be done than simply changing curriculum to reflect a proof and reasoning emphasis. Teacher preparation programs must do their best to change how future teachers engage their students in mathematics. In Andrew (2007), a detailed didactic framework is described which may be used to measure the freedom students have within the classroom to engage in reasoning and proof. In this framework, teachers (whether preservice or practicing) are seen along a continuum from traditional lecturing styles onward towards teaching styles which allow students to actively engage in meaningful mathematical thinking. An educator teaching future teachers mathematics is perceived to be ideal if he/she does some of the following: a) put students in charge of their own learning, b) negotiate students through their own (sometimes erroneous) problem solutions, c) allow students to be active, and d) instigate student-to-student interaction.

Interestingly enough, in the above study, four teachers within the same department are compared and it is revealed that students who take mathematics classes at the same institution can experience radically different teaching styles in the mathematics classes they take. Even though the goal of the department is to have their instructors teach within a similar pedagogical structure, the fact is that there is considerable variance among teachers at the same institution. A related study by Andrew (2006) suggests that instructors often encounter negative feedback from their students who are uncomfortable engaging in mathematics in a radically different way. Administrators should support teachers by anticipating student resistance to new teaching approaches and not reprimanding teachers for negative student evaluations which are often connected with teaching style. Advocating mathematical proof more may help shift how people think and feel about mathematics and better support understanding of the discipline.

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